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Title: An Example Given for Determining Critical Dimensions In Fast-Neutron Reactions

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Source: Russian book, Nekotoryye Voprosy Teorii Yadra, Moscow/Leningrad 1948.

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Sec 24. An Example Given for Determining Critical Dimensions in Fast-Neutron Reactions

Let us consider a sphere consisting of matter in which a chain reaction can be sustained without the participation of thermal neutrons. For simplicity's sake we shall assume that there is no insulation. Let us determine the critical dimensions of the sphere (The authors' ~~NOTE~~: We shall follow here the exposition given by R. Peierls, in Proc Camb Phil 35, 610, 1939.)

The peculiar feature of the problem under examination is that the critical dimensions may prove to be comparable to the length of a neutron's free path. This is connected with two circumstances: first, during fast-neutron reactions there is no need to delay neutrons; secondly, in the case considered the magnification constant k may be much greater than in the case of systems operating with slow neutrons (Authors' ~~NOTE~~: The critical dimensions (linear), are inversely proportional (k - 1)$^{\frac{1}{2}}$, according to a formula derived in Section 23.)

If the dimensions of the system are large in comparison with the free-mean-path of neutrons, it is possible to employ a diffusion picture to describe a chain reaction. In this instance the density of neutrons will satisfy a diffusion equation (see Section 23).

If the dimensions of the system are comparable to the free-mean-path the diffusion theory is inapplicable. In this case it is necessary to proceed from an accurate kinematical equation.

To obtain semi-qualitative results we make the following simplifying assumptions:

1. The mean-free-path of a neutron (allowing for the possibility of fission, radiation capture and scattering) does not depend on neutron energy; in other words the following quantity does not depend on energy

$$\alpha = \frac{1}{l} = N(\sigma_f + \sigma_r + \sigma_s)$$

where the sigmas $\sigma_f, \sigma_r, \sigma_s$ are, respectively, cross sections of fission, radiation capture and neutron scattering and N is the number of nuclei per unit volume.

2. The following expression does not depend on energy

$$\beta = N(\sigma_s + \nu\sigma_f)$$

where nu ν is the number of neutrons originating in one fission process of the nucleus.

3. All neutrons bred have identical energies.

4. There is no non-elastic scattering of neutrons.

5. The elastic scattering of neutrons is spherically symmetrical.

In satisfying these hypotheses instead of the distribution function dependent on both coordinates and neutron's velocity components we can employ the ordinary density of neutrons to describe a neutron-field. We shall denote this density at the point (x, y, z) at a moment of time t by n (x, y, z, t).

The total number of scattered and bred secondary neutrons in an element of volume dx'dy'dz' at time dt' obviously equals:

$$\beta v n(x', y', z', t') \cdot dx'dy'dz'$$

where v is the velocity of the neutrons.

We shall now trace the motion of these neutrons.

At distance r from the point (x', y', z') their number decreases $e^{-\alpha r}$ times. They will uniformly fill a spherical layer of radius r and thickness v·dt'. The density of neutrons at point (x, y, z) of the layer under consideration at a moment of time $t' = t - r/v$ will be:

[226] $$\frac{\beta}{4\pi r^2} n(x', y', z', t') \cdot e^{-\alpha r} \cdot dx'dy'dz' \qquad \text{(a)}$$

$$\text{where } t' = t - r/v.$$

Integrating this expression with respect to (x', y', z'), we shall obviously obtain the density at point (x, y, z) at moment t; that is, the expression n (x, y, z, t):

[227] $$n(x, y, z, t) = \frac{\beta}{4\pi}\int n\left(x', y', z', t - \frac{r}{v}\right) e^{-\alpha r} \cdot \frac{1}{r^2}\, dx'dy'dz' \qquad (24.1)$$

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where integration is effected throughout the volume of the body.

We have obtained on the above assumptions an intergal equation for the density n of neutrons which replaces the general kinetic equation. Let us study this equation.

We shall now look for a solution of (24.1) in the form:

$$n(x,y,z,t) = n(x,y,z)\cdot e^{\lambda t} \qquad (24.2)$$

obviously, values $\lambda > 0$ correspond to the development of a chain reaction.

After substituting (24.2) in it, we obtain (24.1) in the form

[227] $$n(x,y,z) = \frac{\beta}{4\pi}\int n(x',y',z')\cdot e^{-(\alpha+\frac{\lambda}{v})r}\cdot\frac{dx'dy'dz'}{r^2} \qquad (24.3)$$

If the system has critical dimensions, then $\lambda = 0$. Determination of the critical dimensions is thus reduced to the following mathematical problem: It is required to prove under such conditions whether or not the equation

[227] $$n(x,y,z) = \frac{\beta}{4\pi}\int n(x',y',z')\, e^{-\alpha r}\cdot\frac{dx'dy'dz'}{r^2} \qquad (24.3')$$

has non-trivial solutions.

We shall examine the case where the multiplying system is in the shape of a sphere and shall show that there is a non-trivial solution only for a definite critical radius of the sphere.

Let us note that in finding the solution of equation (24.3') we likewise find the solution of the general equation (24.3) simply by substituting $\alpha + \lambda/v$ for α.

We now formulate the conditions governing the behavior of a self-sustained chain reaction. To to this, it is obviously necessary that the number of bred new neutrons per unit length of the neutron's path should exceed the number of absorbed neutrons. The former number equals $N\sigma_f\nu$ and the latter equals $N\sigma_c$, where σ_c is the cross-section of neutron absorption, which equals $\sigma_c = \sigma_f + \sigma_r$.

Now, for a chain reaction we must have

$$N\sigma_f \nu > N\sigma_c$$

or $$(\nu - 1)\sigma_f > \sigma_r \quad . \qquad (24.4)$$

If alpha and beta, α and β, introduced above be employed, the condition necessary for a chain reaction can be presented in the form

$$\beta > \alpha \qquad (24.4')$$

We shall now examine two limiting cases when $\beta - \alpha \ll \beta$ and $\beta \gg \alpha$.

1) if $\beta - \alpha \ll \beta$, many collisions will be required for a material increase in the number of neutrons. Hence, in this case a diffusion consideration of the problem is possible (see the preceding paragraph).

In fulfilling the condition

$$\beta - \alpha \ll \beta$$

the neutron density undergoes a slight change in distances of the order of length of a free-mean-path. Therefore the function $n(x', y', z')$ in the integral (24.3') can be expanded into a power series in $(x' - x)$, $(y' - y)$, $(z' - z)$ and retain terms no greater than the second order. Far from the boundary, the first-order terms during integration with respect to $dx'dy'dz'$, ~~we get~~ become zero and we arrive at the diffusion equation

[228] $$\frac{\beta}{3\alpha^2}\Delta n + \left(\frac{\beta}{\alpha} - 1\right)n = 0 \, . \qquad (24.5)$$

It is easy to feel convinced that the coefficients appearing before Δn and n in (24.5) have the conventional form for a diffusion equation.

In fact, by multiplying equation (24.5) by v and employing the definition of the quantities α and β, we can write this equation in the form:

$$\frac{1}{3} l v \Delta n + v l_c^{-1}(\nu\theta - 1)n = 0$$

[228] (24.5')

where l is the mean-free-path of a neutron with regard to both scattering and capture; $l_c = (N \cdot \sigma_c)^{-1}$ is the path with regard to capture and $\theta = \frac{\sigma_f}{\sigma_c}$.

The quantity $\frac{1}{3} \ell v$ is the coefficient of neutron diffusion and is the lifetime of a neutron relative to capture. The term $(v/\ell_c)n$ gives the number of neutrons absorbed per unit time in a unit volume, and $(v/\ell_c)\nu\theta n$ is the number of neutrons bred per unit time in unit volume.

For a solution of (24.5) it is necessary to know the boundary condition on the outer surface of the multiplying system.

It can be demonstrated [NOTE: See E. Hopf, Cambr Tracts in Math and Math Phys. N-31 (1934), 54, Equation 171.] that for large systems whose dimensions are considerably greater than the mean-free-path of a neutron, this solution takes the form:

[229]
$$0.71 \ell \cdot \frac{\partial n}{\partial x} + n = 0 \qquad (24.6)$$

where x is the direction of the external normal toward the surface of the body.

For large systems we have $\frac{\partial n}{\partial x} \approx \frac{n}{R}$, where, according to order of magnitude, R characterizes the linear dimensions of the system. Hence the ratio of the first term to the second in (24.6), according to the order of magnitude, equals $\ell/R \ll 1$. Thus, in the first approximation the boundary condition for large systems can be formulated as a requirement that neutron density must converge toward zero on the outer surface of a system:

$$n = 0. \qquad (24.6')$$

Since, in the case of small multiplications examined by us, when $\beta - \alpha \ll \beta$, the dimensions of the system are considerably greater than the mean-free-path, we shall use condition (24.6') in the first approximation.

Let us revert to (24.5) and find its solution for the case of a multiplying sphere.

The spherically symmetrical solution of (24.5) has the form:

$$n(r) = \text{const.} \frac{\sin pr}{r} \qquad (24.7)$$

where r is the distance to the center of the sphere and

[230]
$$p = \left[3\alpha^2\left(1 - \frac{\alpha}{\beta}\right)\right]^{1/2}. \qquad (a)$$

If we start from boundary condition (24.6'), we shall find the critical radius R_o from the equation $\sin pR_o = 0$; whence [NOTE: Neutron density is essentially positive. Hence we must take the smallest non-zero root of the equation $\sin x = 0$.]:

[230] $$R_o = \frac{\pi}{p} = \frac{\pi}{\alpha}\left[3\left(1-\frac{\alpha}{\beta}\right)\right]^{-\frac{1}{2}} = \pi\left[\frac{\frac{1}{3}\lambda\lambda_c}{\nu\theta - 1}\right]^{\frac{1}{2}} = \pi\left[\frac{D\tau_c}{k-1}\right]^{\frac{1}{2}} \qquad (24.8)$$

where $\ell = \frac{1}{\alpha} = [N(\sigma_s + \sigma_c)]^{-1}$ is the mean-free-path of a neutron relative to scattering and absorption; $\ell_c = 1/N\sigma_c$ is the mean-free-path of a neutron relative to absorption; $\theta = \sigma_f/\sigma_c$; $D = \frac{1}{3}\ell v$ is the magnification constant of neutrons; $\tau_c = \ell_c/v$ is the life time of a neutron relative to capture; $k = \nu\theta$ is the neutron multiplication constant.

Since the condition is $1 - \alpha/\beta \ll 1$, it follows from (24.8) that the critical radius is appreciable greater than the free path ℓ. This is corroborated by preceding statements.

It should be noted that formula (24.8) coincides with formula (23.16) for the critical dimensions of a cube if the length of neutron retardation r_o in the latter is assumed equal to zero and subtracted from the multiplier $\sqrt{3}$ connected with another geometrical problem. This results from the fact that in both cases we utilized a diffusion viewpoint, which holds good for small multiplications.

The use of the exact boundary condition (24.6) leads to the following equation for the determination of the value R_o:

[230] $$pR_o \cdot \cot pR_o = 1 - \frac{\alpha R_o}{0.71} \qquad (24.9)$$

Denoting $1 - \alpha/\beta$ by ξ^2 and disregarding any higher powers of xi ξ than the second power when the ξ's are small, we shall obtain from (24.9):

[231] $$\frac{1}{\beta R_o} = \frac{\sqrt{3}}{\pi}\xi + 0.71\frac{3}{\pi^2}\xi^2 = 0.55\xi + 0.22\xi^2. \qquad (24.10)$$

2) Now let us examine the other limiting case, namely that of large multiplications, when $\beta \gg \alpha$.

Let us note that this is a fictitious and practically non-existent case, since in order to satisfy the condition $\beta \gg \alpha$, then nu ν must be considerably greater than unity. Nevertheless, it is interesting to examine this case because thereby we obtain critical dimensions considerably smaller than the mean-free-path l.

Given the dependence of the critical radius on the effective cross-sections in the two limiting cases $R_o \gg l$ and $R_o \ll l$, one can find R_o through interpolation in the intermediate region ($R_o \approx l$) for which it is difficult to find a direct expression of the critical dimensions.

In satisfying the condition $\beta \gg \alpha$ we can disregard the exponent of the factor $e^{-\alpha R}$ which forms part of integral equation (24.1), since it appears that the radius of the sphere in the case under consideration is appreciably smaller than the mean-free-path.

Thus, when $\beta \gg \alpha$, we obtain from (24.3') equation

[231]
$$n(x, y, z) = \frac{\beta}{4\pi} \int n(x', y', z') \frac{dx'dy'dz'}{r^2} \tag{24.11'}$$

where $r^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2$.

We shall assume that the neutron density n depends only on the distance ρ to the center of the sphere. Then it is easy to carry out integration by angles in (24.11'). We obtain as a result:

[231]
$$n(\rho) = \frac{\beta}{2} \int_0^R \frac{\rho'}{\rho} n(\rho') \cdot \ln \frac{\rho + \rho'}{|\rho - \rho'|} d\rho' \tag{24.11}$$

where R is the radius of the sphere.

Introducing the function $f(\rho) = n(\rho)$ instead of $n(\rho)$, and denoting p/R by x, we can rewrite (24.11) as

[232]
$$\frac{2}{\beta R} f(x) = \int_0^1 f(x') \cdot \ln \frac{x + x'}{|x - x'|} \cdot dx' \tag{24.12}$$

$$f(0) = 0.$$

or as

$$\lambda f = Lf, \; f(0) = 0, \qquad (24.12')$$

where $\lambda = 2/\beta R$ and L is the integral operator appearing in (24.12).

We have now obtained an integral equation with a positive symmetrical kernel. The point which interests us is the greatest eigenvalue λ_0 for this equation.

For an approximate expression for λ_0 we shall start from the following property of the greatest eigenvalue. Let f(x) be some function which does not admit negative values and which is characterized by the property that the ~~relation~~ ratio Lf/f is finite. Now, if Lf/f is comprised between the limits Δ_1 and Δ_2:

$$\Delta_1 < Lf/f < \Delta_2$$

then the greatest eigenvalue λ_0 will also be found within these limits, where $\Delta_1 < \lambda_0 < \Delta_2$. [NOTE: It is easy to prove this in the following manner. If $\phi_0(x)$ is an eigenfunction of an operator L corresponding to the largest eigenvalue λ_0:

$$L\,\phi_0(x) = \lambda_0 \cdot \phi_0(x), \qquad \mathrm{I}$$

then the integral

[232]

$$\int_0^1 f(x)\,\phi_0(x)\cdot dx \qquad \mathrm{II}$$

cannot be equal to zero since the integrand function does not change its sign. In fact, in accordance with the conditions, the function f(x) only assumes positive values and $\phi_0(x)$ as an eigenfunction corresponding to the largest eigenvalue does not converge toward zero at any point and consequently does not change its sign.

Let us expand the function f(x) into a series in eigenfunctions of the operator L (expanded in the order of decreasing eigenvalues):

[233]

$$f(x) = \sum_{j=0}^{\infty} C_j \phi_j(x). \qquad \mathrm{III}$$

Because the above equation (II) cannot equal zero, the coefficient C_0 cannot equal zero:

$$C_0 \neq 0.$$

Applying n times the operator L to both sides of III equation, we obtain:

[233] $$L^n f(x) = \sum_{j=0}^{\infty} C_j \lambda_j^n \Phi_j(x) \qquad \text{(a)}$$

whence $$L^n f(x)/\lambda_0^n = \sum_{j=0}^{\infty} C_j \left(\frac{\lambda_j}{\lambda_0}\right)^n \Phi_j(x)$$

[233] (b)

As n tends toward infinity, all terms except the first on the right side of the latter equation tend toward zero.

Hence $$\lim_{n\to\infty} L^n f(x)/\lambda_0^n = C_0 \Phi_0(x)$$

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On the other hand the condition $\Lambda_1 < L < \Lambda_2$ can be rewritten in the form

$$\Lambda_1 f < Lf < \Lambda_2 f \, .$$

Applying k times the operator L to this inequality, we obtain:

$$\Lambda_1 L^k f < L^{k+1} f < \Lambda_2 L^k f \, .$$

From this follows

$$\Lambda_1 < L^{k+1} f / L^k f < \Lambda_2$$

and consequently $$\Lambda_1^k < \Lambda^k f / f < \Lambda_2^k$$

In particular, we have: $$\frac{\Lambda_1^n}{\lambda_2^n} f < \frac{L^n f}{\lambda_0^n} < \frac{\Lambda_2^n}{\lambda_0^n} f$$

From this and from (IV) it follows that

$$\lim_{n\to\infty} (\Lambda_1/\lambda_0)^n f < C_0 \Phi_0(x) < \lim_{n\to\infty} (\Lambda_2/\lambda_0)^n f \, ;$$

these inequalities are possible only in case

$$\Lambda_1 < \lambda_0 < \Lambda_2 \qquad \text{Q.E.D.}$$

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Whence it follows that if we succeed in selecting a function $f(x)$ (which does not admit negative values) such that its magnitudes Λ_1 and Λ_2 differ only slightly from each other, we shall thereby find λ_0 with a great degree of accuracy.

We shall take as $f(x)$ the function

$$f(x) = x - bx^3, \tag{24.13}$$

where b is a constant properly selected.

It is easy to see that

$$[233] \quad Lf = \frac{1}{2}(1-x^2)\ln\frac{1+x}{1-x} + x - b\left\{\frac{1}{4}(1-x^2)\ln\frac{1+x}{1-x} + \frac{x^3}{2} + \frac{x}{6}\right\} \quad \text{(a)}$$

For $b = 0$, Lf/f varies from 1 to 2; these values are reached at the points $x = 0$ and $x = 1$.

Let us select b in such a manner that, at the limits of the interval (0, 1), the function $f(x)$ will receive identical values; for this purpose b must equal 0.639. If we assume function (24.13) to have this value of b, it turns out that

$$\Lambda_1 = 1.55 \quad \text{and} \quad \Lambda_2 = 1.59 .$$

Whence it follows that

$$\lambda_0 = 1.57 \pm 0.02 .$$

Now, when $\beta \gg \alpha$, the critical radius R equals:

$$R_0 = \frac{2}{1.57\,\beta} = \frac{1}{0.78\,\beta}$$

We see that in the case of great multiplications, $\beta \gg \alpha$, the critical dimensions prove to be less than the mean-free-path ℓ - $1/\alpha$ of a neutron.

[235]

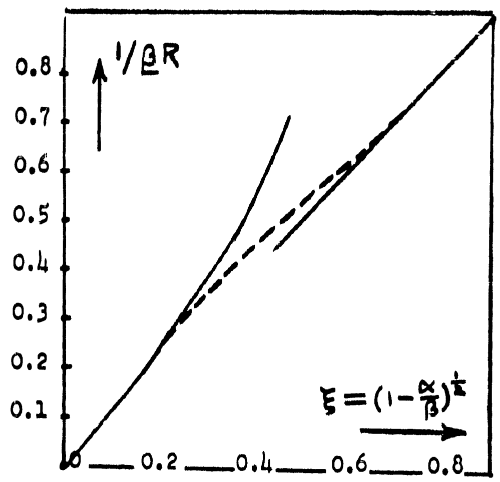

Figure 8.

The critical dimensions for intermediate values of the ratio β/α can be found by interpolation; see Figure 8 which shows the dependence of $1/\beta \cdot R$ upon xi $\xi = (1 - \alpha/\beta)^{1/2}$.

- End of Section 24 -

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Some Problems of Nuclear Theory
by A. Akhiyezer & I. Pomeranchuk

(Nekotoryye Voprosy Teorii Yadra)
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